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PHIL309P

Problem Set 3

1. Case: Condorcet’s Rule. Condorcet’s rule, known more familiarly as majority rules, states that if more people prefer A over B than B over A, then the group prefers A over B. Starting with Group Rationality (all preferences are transitive and complete) we see an immediate contradiction.

|  |  |  |  |
| --- | --- | --- | --- |
|  | Voter 1 | Voter 2 | Voter 3 |
| Pref 1 | A | B | C |
| Pref 2 | B | C | A |
| Pref 3 | C | A | B |

In the example above if we were to apply Condorcet’s Rule we come up with the preference ranking of A > B, B > C, and C > A. While this does satisfy the completeness component of Group Rationality where each element has some type of relation or preference order to each other element, it does not satisfy transitivity. The contradiction arises because we want to conclude that since (A > B) && (B > C) then (A > C) which is not the case, the preferences derived from Condorcet’s Rule yield a cycle and cycles are intransitive. Moving on to the second axiom, Universal Domain states that the domain of the social ordering function must be the set of all preference rankings, or in simpler terms that every voter’s preference ranking is taken into consideration. Condorcet’s Rule complies with Universal Domain because to determine if the majority of the voters prefer one candidate over another every voter’s preference ranking must be surveyed. In terms of Pareto, Condorcet’s Rule holds up. According to Pareto if all voters rank A over B then the group’s ranking is A over B. The reason this works is because if everyone strictly prefers A to B then we can without a doubt conclude that A has the majority of preferences. The last axiom that we need to address is the Independence of Irrelevant Alternatives. Independence of Irrelevant Alternatives states that when deciding whether A is preferred over B, information about the voters’ preference for C should have no effect. This axiom is upheld by Condorcet. Going back to the original example, even if you were to move C up to pref 2 or even up to pref 1 A and B would keep their preference relation and the group’s preference relation for A and B would remain untouched.

1. Case: Borda Count. Analyzing the group ranking of a Borda Count in terms of transitivity and completeness, the Group Rationality axiom of Arrow’s Theorem, we see that it satisfies this axiom. Borda Count works by assigning each voters’ preferences a value based on the ranking the voter gave it and in terms of a group ranks each based on their overall score. This method accomplishes transitivity because the Borda Count of the group’s top will be greater than or equal to the second which is greater than or equal to the Borda Count of the last preference ranking. Because the rankings are based on a utility function each ranking, looking at them in order, is greater than or equal to the next. We can conclude transitivity because if the Borda Count of pref 1 is greater than that of pref 2 (disregarding indifference) then it follows that the ranking of pref 1 is greater than that of pref 3. Transitivity allows us to develop a similar relation for all the candidates being considered, in other words each element has a relation to any of the other elements, the definition of a complete set finishing the proof of Group Rationality for a Borda Count Group Ranking system. For the Universal Domain axiom of Arrow’s Theorem, we can apply the same logic that was used for Condorcet’s Rule. The group preference ranking determined by Borda Count is based on the collective utility of each candidate. In calculating this utility, Borda Count looks at every voter’s preference ranking before determining the group’s ranking. This is the very definition of Universal Domain so we can conclude that the axiom is satisfied. Moving on to the Pareto axiom, which to restate is that if all voters prefer candidate A over candidate B then the group prefers A over B, we can see that this is upheld by Borda Count. If every voter were to rank A over B then in everyone’s Borda Count A would receive more points than B which when all Borda Counts are summed results in A maintaining a higher count than B. The last axiom in Arrow’s Theorem, Independence of Irrelevant Alternatives, is where Borda Count fails. Let’s look at the following example:

|  |  |  |
| --- | --- | --- |
|  | Voter 1 | Voter 2 |
| Pref 1 – 3 points | A | B |
| Pref 2 – 2 points | B | C |
| Pref 3 – 1 point | C | A |

I’ll use a 3-2-1-point system for preference rankings 1, 2, and 3 respectively for this example. In this scenario candidate A has a Borda Count of 4, B has a count of 5, and C has 3 giving a group preference ranking of B > A > C. Let’s look what happens when, for some reason, voter 1 decides that they prefer candidate C over B, and voter 2 decides that they now prefer A over C.

|  |  |  |
| --- | --- | --- |
|  | Voter 1 | Voter 2 |
| Pref 1 – 3 points | A | B |
| Pref 2 – 2 points | C | A |
| Pref 3 – 1 point | B | C |

The new group ranking becomes A (5 points) > B (4 points) > C (3 points). The reason this violates IIA is because even though voters 1 and 2 don’t change their preference between A and B, the group preference relation between A and B is affected.

1. Case: Plurality Ranking. Plurality Ranking, which is like Borda Count, rewards each candidate 1 point per first place vote and 0 for preference 2 and 3. Because of its close relation to Borda Count, we can use the same rationale for why it upholds the Group Rationality axiom. If we look at the group’s preference ranking in order from top to bottom, the candidate’s total utility will be greater than or equal to the one below it. This prevents the possibility of a cycle appearing because the only relation where both ARB and BRA both appear in the group’s set of preferences is indifference (if their cumulative utilities are equal). If that isn’t the case then the total utility of the top preference will be greater than the second and regardless of the second preference’s relation to the third preference the top reference will have a higher utility than the third giving the group’s set of preferences transitivity. If a preference set is transitive we can conclude that it is also complete because it is a set of relative relations so by transitivity every candidate will have a relative preference to every other candidate being considered completing the requirements for Arrow’s Group Rationality axiom. Universal Domain is trivial now that we already proved it for Borda Count. The way in which a candidate is ranked depends on the total number of first place votes they receive meaning that every preference ranking must be analyzed to assign the candidate a value. The next axiom we must analyze is Pareto. Plurality works for Pareto because if every voter prefers A over B then B won’t win the election. In a Plurality ranking system there are 2 situations that can occur when A is preferred over B, the first being if A is ranked first and the second is if A is not ranked first. In both scenarios, B is any candidate that A is preferred over. In both cases, there is no way in which B can win the election because the only preference rank that is rewarded a point is the top preference and under Pareto A is strictly preferred to B for all voters. The last axiom is Independence of Irrelevant Alternatives and this is where Plurality, like Borda Count, fails. Let’s start with the following preference rankings, preference ranking on the left and the number of voters with each ranking on the top:

|  |  |  |  |
| --- | --- | --- | --- |
|  | 4 Voters | 3 Voters | 2 Voters |
| Pref 1 – 1 point | A | B | C |
| Pref 2 – 0 points | B | D | B |
| Pref 3 – 0 points | C | C | A |
| Pref 4 – 0 points | D | A | D |

Based on these preferences we have a group preference ranking of A (4 points) > B (3 points) > C (2 points) > D (0 points), and A wins the election. Now imagine some scenario where candidate C decides to drop out prior to the election leaving us with:

|  |  |  |  |
| --- | --- | --- | --- |
|  | 4 Voters | 3 Voters | 2 Voters |
| Pref 1 – 1 point | A | B | B |
| Pref 2 – 0 points | B | D | A |
| Pref 3 – 0 points | D | A | D |

The new group preference ranking is B (5 points) > A (4 points) > D (0 points) and B now wins the election. This is not in line with IIA because the group preference relation between A and B flips based on the actions of a third candidate C.

1. Case: A Reverse Dictatorship. In a Reverse Dictatorship, the group’s preference ranking is determined by taking the reverse preference order of some voter, I’ll call him Fidel, and disregards the preference rankings of the other voters. Let’s say that Fidel has the preference order A > B > C, the group’s preference order would then be C > B > A. As we can see there are no cycles in this type of social order function because Fidel can only rank each candidate once. The preference relation C > A can now be derived because it doesn’t cause a contradiction in the set of binary preference relations of the group. As in other social order functions, this gives us the preference relations to conclude that the social order function is also complete and that a reverse dictatorship satisfies Group Rationality. The fault of a reverse dictatorship comes to light when we look at Arrow’s Universal Domain axiom. The Universal Domain axiom states that the domain of the social order function is the set of all profiles. When dealing with a reverse dictatorship the domain is just Fidel’s profile because the group ranking can be determined solely by his personal preference ranking contradictory to Universal Domain. Unlike a normal dictatorship, a reverse dictatorship also comes up short when looking at the Pareto axiom. According to Pareto, if everyone prefers A over B then this should be reflected in the group’s preference ranking. Let’s assume that every voter has the preference of A over B. If this was a dictatorship then this wouldn’t be an issue because then the group preference would also be A over B, because that’s what Fidel preferred also, but by the definition of a reverse dictatorship the group’s preference would be B over A causing the condition for the Pareto axiom to be false. Lastly, a reverse dictatorship satisfies Independence of Irrelevant Alternatives. The fact that we only need to look at a single voter’s preference ranking simplifies the proof for IIA in a reverse dictatorship. Given the relationship between A and B, if Fidel was to switch his preference regarding a third candidate, or that candidate was no longer part of the election, the relationship between A and B wouldn’t change in the group’s preference ranking.
2. Case: Unanimity. The Unanimity social welfare function says that if all voters have the same preference ranking then that is the preference ranking for the group, otherwise all candidates are tied. For each of the axioms there are 2 cases we must look at and a third for only some of the axioms: 1) all the voters have the same preference ranking, 2) there is a single voter who has a different preference ranking than the rest of the voters, and 3) more than one voter has different a different preference ranking. Unanimity passes transitivity for both cases 1 and 2 stated above. If everyone has the same preference ranking then that ranking is the groups preference ranking. The same logic for a dictator’s preference can be applied here so there won’t be any cycles and by after clarifying that we can derive transitivity and completeness for Group Rationality, which also is proven true if every candidate is tied because the group is indifferent for every candidate. Unanimity also satisfies Universal Domain because each voter’s ranking is analyzed to determine if they all have the same preference rank. Unanimity and Pareto are also closely related. If everyone ranks A over B, where A and B are any 2 candidates being considered, then the group ranking will be that ranking and the candidates won’t be tied. Finally, we are left with IIA. IIA is where unanimity fails. Let’s look at the scenario where everyone has the preference ranking of A > B > C, and more specifically the relation of preferring A over B. If any of the voters decide to change the relation between A and C or B and C, even though they all still prefer A over B, the new group ranking is A ~ B ~ C and A is no longer preferred over B.